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Order $1/N^2$ test of the Maldacena conjecture II: the full bulk one-loop contribution to the boundary Weyl anomaly

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Abstract

We compute the complete bulk one-loop contribution to the Weyl anomaly of the boundary theory for IIB supergravity compactified on $AdS_5 \times S^5$. The result, that $\delta\mathcal{A} = (E+I)/\pi^2$, reproduces the subleading term in the exact expression $\mathcal{A} = -(N^2-1)(E+I)/\pi^2$ for the Weyl anomaly of $\mathcal{N}=4$ super-Yang–Mills theory, confirming the Maldacena conjecture. The anomaly receives contributions from all multiplets casting doubt on the possibility of describing the boundary theory beyond leading order in N by a consistent truncation to the 'massless' multiplet of IIB supergravity.

Henningson and Skenderis' beautiful computation [1] of the Weyl anomaly of $\mathcal{N}=4$ $SU(N)$ super-Yang–Mills theory from five-dimensional gravity is a remarkable test of the Maldacena conjecture [2] to leading order in large N. When super-Yang–Mills theory is coupled to a nondynamical, external metric, g_{ij}, the Weyl anomaly, $\mathcal{A}$, is the response of the logarithm of the partition function, F, to a scale transformation of that metric: $\delta F = \int d^4x\,\sqrt{g}\,\delta\sigma\,\mathcal{A}$ when $\delta g_{ij} = 2\delta\sigma\, g_{ij}$. On general grounds $\mathcal{A} = aE + cI$ where E is the Euler density, $(R^{ijkl}R_{ijkl} - 4R^{ij}R_{ij} + R^2)/64$, and I is the square of the Weyl tensor, $I = (-R^{ijkl}R_{ijkl} + 2R^{ij}R_{ij} - R^2/3)/64$. A one-loop calculation [3] gives $\mathcal{A}$ as the sum of contributions from the six scalars, two fermions and gauge vector of the super-Yang–Mills theory (all in the adjoint with dimension N^2-1)

$$\mathcal{A} = \frac{(6s + 2f + g_v)(N^2-1)}{16\pi^2}. \tag{1}$$

When the heat-kernel coefficients s, f, and g_v are expressed in terms of E and I this becomes

$$\mathcal{A} = -\frac{(N^2-1)(E+I)}{\pi^2}, \tag{2}$$

so $a = c = -(N^2-1)/(2\pi^2)$ and supersymmetry protects this from higher-loop corrections. Henningson and Skenderis showed that the tree-level calculation in the bulk reproduces the leading N^2 piece by solving the Einstein equations perturbatively near the boundary. We would expect that the -1 piece is due to string loops in the bulk that to this order can be approximated by field theory loops, but these depend on much more

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than just classical general relativity, and reproducing them provides a more stringent test of the Maldacena conjecture sensitive to the detailed particle content of the bulk IIB supergravity theory. In [4] we showed that the bulk supergravity one-loop contributions to $a-c$ vanished when summed over each supermultiplet confirming the conjecture. In this Letter we will complete this calculation of the Weyl anomaly by computing a itself and showing that it does indeed reproduce the -1 piece.

The one-loop contribution to $\mathcal{A}$ from bulk fields was found in [5] using Schrödinger functional methods that are particularly appropriate to the AdS/CFT correspondence because, being Hamiltonian, they apply four-dimensional technology to the study of fields on a five-dimensional manifold with a boundary. The result can be expressed [6] as

$$\delta\mathcal{A} = -\sum \frac{(\Delta-2)a_2}{32\pi^2}, \tag{3}$$

where the sum is taken over all the fields in IIB supergravity compactified on $AdS_5 \times S^5$, Δ is the scaling dimension of the associated boundary operator, and a_2 is a four-dimensional heat-kernel coefficient (multiplied by -1 for anticommuting fields). Deriving this requires decomposing the five-dimensional components of fields into those appropriate to the four-dimensional boundary.

In deriving (3) the AdS metric was taken to be

$$ds^2 = \frac{1}{t^2}\left(l^2\,dt^2 + \sum_{i,j} \hat{g}_{ij}\,dx^i\,dx^j\right), \quad t > 0 \tag{4}$$

which satisfies the Einstein equations with cosmological constant $-6/l^2$ provided $\hat{g}_{ij}$, (which is proportional to the boundary metric), is Ricci flat. In this case $E=-I$ so that $\mathcal{A}$ is proportional to $a-c$. To find a itself it is convenient to take a constant curvature boundary for which $R_{ijkl} = (g_{ik}g_{jl} - g_{il}g_{jk})R/12$, $R_{ij} = Rg_{ij}/4$, $I=0$ and $E = R^2/384$. The solution to Einstein's equations is obtained by multiplying $\hat{g}_{ij}$ in (4) by $(1-\widehat{R}t^2l^2/48)^2$, where $\widehat{R}$ is the curvature constructed from $\hat{g}_{ij}$. The effect of this extra piece on the decomposition of five-dimensional fields into four-dimensional variables is to introduce into the four-dimensional operators precisely those couplings to $\widehat{R}$ that render them conformally covariant. Thus a_2 for a five-dimensional gauge field is the heat-kernel coefficient for the operator associated with a four-dimensional gauge field, whilst that for a minimally coupled five-dimensional scalar is associated with a conformally coupled four-dimensional scalar.

The scaling dimensions Δ are related to the bulk masses which were originally worked out in [7]. In Table 1 we display the corresponding values of $\Delta - 2$. The multiplets are labeled by an integer $p \geqslant 2$, and the fields form representations of $SU(4) \sim SO(6)$. The four-dimensional heat-kernel coefficients have also been known for a long time and we use the values given by [8,9]. In Table 2 we list these for the cases of a Ricci flat boundary.

Table 1
Mass spectrum. The supermultiplets (irreps of $U(2,2/4)$) are labeled by the integer p. Note that the doubleton ($p=1$) does not appear in the spectrum. The (a,b,c) representation of $SU(4)$ has dimension $(a+1)(b+1)(c+1)(a+b+2)(b+c+2)(a+b+c+3)/12$, and a subscript c indicates that the representation is complex. (Spinors are four component Dirac spinors in AdS_5)

Field	$SO(4)$ rep$^{\mathrm{n}}$	$SU(4)$ rep$^{\mathrm{n}}$	$\Delta - 2$
$\phi^{(1)}$	$(0,0)$	$(0,p,0)$	$p-2, \quad p\geqslant 2$
$\psi^{(1)}$	$(\frac{1}{2},0)$	$(0,p-1,1)_c$	$p-3/2, \quad p\geqslant 2$
$A^{(1)}_{\mu\nu}$	$(1,0)$	$(0,p-1,0)_c$	$p-1, \quad p\geqslant 2$
$\phi^{(2)}$	$(0,0)$	$(0,p-2,2)_c$	$p-1, \quad p\geqslant 2$
$\phi^{(3)}$	$(0,0)$	$(0,p-2,0)_c$	$p, \quad p\geqslant 2$
$\psi^{(2)}$	$(\frac{1}{2},0)$	$(0,p-2,1)_c$	$p-1/2, \quad p\geqslant 2$
$A^{(1)}_{\mu}$	$(\frac{1}{2},\frac{1}{2})$	$(1,p-2,1)$	$p-1, \quad p\geqslant 2$
$\psi^{(1)}_{\mu}$	$(1,\frac{1}{2})$	$(1,p-2,0)_c$	$p-1/2, \quad p\geqslant 2$
$h_{\mu\nu}$	$(1,1)$	$(0,p-2,0)$	$p, \quad p\geqslant 2$
$\psi^{(3)}$	$(\frac{1}{2},0)$	$(2,p-3,1)_c$	$p-1/2, \quad p\geqslant 3$
$\psi^{(4)}$	$(\frac{1}{2},0)$	$(0,p-3,1)_c$	$p+1/2, \quad p\geqslant 3$
$A^{(2)}_{\mu}$	$(\frac{1}{2},\frac{1}{2})$	$(1,p-3,1)_c$	$p, \quad p\geqslant 3$
$A^{(2)}_{\mu\nu}$	$(1,0)$	$(2,p-3,0)_c$	$p, \quad p\geqslant 3$
$A^{(3)}_{\mu\nu}$	$(1,0)$	$(0,p-3,0)_c$	$p+1, \quad p\geqslant 3$
$\psi^{(2)}_{\mu}$	$(1,\frac{1}{2})$	$(1,p-3,0)_c$	$p+1/2, \quad p\geqslant 3$
$\phi^{(4)}$	$(0,0)$	$(2,p-4,2)$	$p, \quad p\geqslant 4$
$\phi^{(5)}$	$(0,0)$	$(0,p-4,2)_c$	$p+1, \quad p\geqslant 4$
$\phi^{(6)}$	$(0,0)$	$(0,p-4,0)$	$p+2, \quad p\geqslant 4$
$\psi^{(5)}$	$(\frac{1}{2},0)$	$(2,p-4,1)_c$	$p+1/2, \quad p\geqslant 4$
$\psi^{(6)}$	$(\frac{1}{2},0)$	$(0,p-4,1)_c$	$p+3/2, \quad p\geqslant 4$
$A^{(3)}_{\mu}$	$(\frac{1}{2},\frac{1}{2})$	$(1,p-4,1)$	$p+1, \quad p\geqslant 4$

Table 2
Anomaly coefficients of massive fields on AdS_5. Note that the massive vector coefficient is $v_0 + 2s - 2s_0$ where v_0, s, s_0 are respectively, the coefficients for the 4d gauge-fixed Maxwell operator, a conformally coupled scalar, and a minimally coupled scalar

Field	$R_{ij} = 0$: $180a_2/R_{ijkl}R^{ijkl}$	Constant R: $180a_2/R^2$
ϕ	1	$-1/12$
ψ	$7/2$	$-11/12$
A_μ	-11	$29/3$
$A_{\mu\nu}$	33	$19/4$
ψ_μ	$-219/2$	$-61/4$
$h_{\mu\nu}$	189	$747/4$

If we denote the values of a_2 for the fields ϕ, ψ, A_μ, $A_{\mu\nu}$, ψ_μ, $h_{\mu\nu}$ by $s, f, v, a, r,$ and g respectively then the contribution from a generic ($p \geqslant 4$) multiplet is

$$\Big(\sum(\Delta - 2)a_2\Big)_{p\geqslant 4} = (-4s + 4a + r + f + 2v)\frac{p}{3} + (-105s - g - 26a - 8r - 72f - 48v)\frac{p^3}{12} + (16v + 20f + 10a + 4r + 25s + g)\frac{p^5}{12} \quad (5)$$

whilst for the $p = 3$ multiplet it is

$$\Big(\sum(\Delta - 2)a_2\Big)_{p=3} = 244f + 18g + 266s + 218v + 148a + 64r. \quad (6)$$

The $p = 2$ multiplet contains gauge fields requiring the introduction of Faddeev–Popov ghosts. Their parameters are given in Table 3 along with the decomposition of the five-dimensional components of fields into four-dimensional pieces.

$$12v - 30s + 6r - 10f + 2g \quad (7)$$

and if we include the scalars, spinors and antisymmetric tensors the total contribution of the $p = 2$ multiplet is

$$\Big(\sum(\Delta - 2)a_2\Big)_{p=2} = 12v - 6s + 6r + 6f + 2g + 12a. \quad (8)$$

Substituting the values of the heat kernel coefficients for a Ricci flat boundary shows that the contribution of each supermultiplet vanishes implying that $a = c$ [4]. However if we do not specialize to this case we have to deal with the sum over multiplets labeled by p. We will evaluate this divergent sum by weighting the contribution of each supermultiplet by z^p. The sum can be performed for $|z| < 1$, and we take the result to be a regularization of the weighted sum for all values

Table 3
Decomposition of gauge fields for the massless multiplet

Original field	Gauge fixed fields	$\Delta - 2$	$R_{ij} = 0$: $180a_2/R_{ijkl}R^{ijkl}$	Constant R: $180a_2/R^2$
A_μ	A_i	1	-11	$29/3$
(**15** of $SU(4)$)	A_0	2	1	$-1/12$
	$b_{\text{FP}}, c_{\text{FP}}$	2	-1	$1/12$
ψ_μ	ψ_i^{irr}	$3/2$	$-219/2$	$-61/4$
	$\gamma^i\psi_i$	$5/2$	$7/2$	$-11/12$
(**4** of $SU(4)$)	ψ_0	$5/2$	$7/2$	$-11/12$
	$\lambda_{\text{FP}}, \rho_{\text{FP}}$	$5/2$	$-7/2$	$11/12$
	σ_{GF}	$5/2$	$-7/2$	$11/12$
$h_{\mu\nu}$	h_{ij}^{irr}	2	189	$727/4$
($SU(4)$ singlet)	h_{0i}	3	-11	$29/3$
	h_{00}, h_μ^μ	$\sqrt{12}$	1	$-1/12$
	$B_0^{\text{FP}}, C_0^{\text{FP}}$	$\sqrt{12}$	-1	$1/12$
	$B_i^{\text{FP}}, C_i^{\text{FP}}$	3	11	$-29/3$

of z. Multiplying this by $1/(z-1)$ and integrating around the pole at $z=1$ gives a regularization of the original divergent sum. This yields

$$\sum(\Delta-2)a_2 = 8s + 4f + 2v \tag{9}$$

which remarkably depends only on the heat-kernel coefficients of fields in the super-Yang–Mills theory. By decomposing a five-dimensional vector into longitudinal and transverse pieces and solving the Schrödinger equation for them, it can be seen that the heat-kernel coefficient for a vector field, v, is related to that for the four-dimensional (gauge-fixed) Maxwell operator, v_0, as $v = v_0 + 2s - 2s_0$ where s_0 is the coefficient for a minimally coupled four-dimensional scalar (Faddeev–Popov ghost), showing $v - 2s = v_0 - 2s_0 = g_v$ [10]. Therefore we finally arrive at the one-loop contribution to the Weyl anomaly

$$\delta\mathcal{A} = -\sum\frac{(\Delta-2)a_2}{32\pi^2} = -\frac{6s+2f+g_v}{16\pi^2} \tag{10}$$

which is precisely what is needed to reproduce the subleading term in the exact Weyl anomaly of super-Yang–Mills theory and verify the Maldacena conjecture.

It is worth emphasizing that a received nontrivial contributions from all the supermultiplets, not just the $p=2$ multiplet containing gauge fields, in contrast to [11]. This indicates that although bulk tree-level solutions might be constructed by a 'consistent' truncation of the full IIB supergravity to this single multiplet, as in studies based on gauged $\mathcal{N}=8$ supergravity, such a procedure would miss loop effects in the bulk that contribute to the super-Yang–Mills theory at subleading order. So, for example, the application of (3) to the spectrum of [12] fails to produce the expected subleading correction to the coefficient c for the infra-red fixed point of the RG flow driven by adding certain mass terms to the $\mathcal{N}=4$ super-Yang–Mills theory to break the supersymmetry down to $\mathcal{N}=1$.

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